

ON TWO NONINTEGRABLE CASES OF THE GENERALIZED HÉNON–HEILES SYSTEM WITH AN ADDITIONAL NONPOLYNOMIAL TERM

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Abstract

The generalized Hénon–Heiles system with an additional nonpolynomial term is considered. In two nonintegrable cases new two-parameter solutions have been obtained in terms of elliptic functions. These solutions generalize the known one-parameter solutions. The singularity analysis shows that it is possible that three-parameter single-valued solutions exist in these two nonintegrable cases. The knowledge of the Laurent series solutions simplifies search of the elliptic solutions and allows to automatize it.

1 INTRODUCTION

Beginning from papers [1–3], investigations of two-dimensional Hamiltonian systems with polynomial potentials attract large attention due to detect of the "dynamical chaos" phenomena. There is no method to find the multivalued general solution of a two-dimensional nonintegrable system in the analytic form. At the same time it is an actual problem to find single-valued special solutions in the analytic form, because the investigation of the solutions with some additional properties, for example, periodic solutions, plays an important role in the study of physical phenomena. Another problem is to pick out nonintegrable cases, in which single-valued special solutions can depend on maximal number of arbitrary parameters.

The Hénon–Heiles Hamiltonian [2]:

$$H = \frac{1}{2}(x_t^2 + y_t^2 + x^2 + y^2) + x^2y - \frac{1}{3}y^3$$

and its generalizations are one of the most actively studied two-dimensional Hamiltonians (see [4] and references therein). The generalized Hénon–Heiles system is a model widely used in astronomy [5] and physics, for example, in gravitation [6, 7].

One of lines of investigation of this system is the search for special solutions [8–13]. The general solutions in the analytic form are known only in the integrable cases [14–17], in other cases not only four-, but even three-parameter exact solutions have yet to be found. In [12] new type of one-parameter elliptic solutions has been obtained. Such solutions exist only in integrable cases and two nonintegrable ones. In these nonintegrable cases there exist three-parameter Laurent-series solutions [18], which generalize the Laurent series of one-parameter elliptic solutions. In this paper we find elliptic two-parameter solutions, which generalize solutions obtained in [12].

2 BASIC EQUATIONS

The generalized Hénon–Heiles system with an additional nonpolynomial term is described by the Hamiltonian

$$H = \frac{1}{2}(x_t^2 + y_t^2 + \lambda_1 x^2 + \lambda_2 y^2) + x^2 y - \frac{C}{3} y^3 + \frac{\mu}{2x^2} \quad (1)$$

and the corresponding system of the motion equations:

$$\begin{cases} x_{tt} = -\lambda_1 x - 2xy + \frac{\mu}{x^3}, \\ y_{tt} = -\lambda_2 y - x^2 + Cy^2, \end{cases} \quad (2)$$

where $x_{tt} \equiv \frac{d^2 x}{dt^2}$ and $y_{tt} \equiv \frac{d^2 y}{dt^2}$, λ_1 , λ_2 , μ and C are arbitrary numerical parameters. Note, that if $\lambda_2 \neq 0$, then one can put $\lambda_2 = \text{sign}(\lambda_2)$ without the loss of generality.

Due to the Painlevé analysis [19–21] the following integrable cases have been found [22]:

- (i) $C = -1$, $\lambda_1 = \lambda_2$,
- (ii) $C = -6$, λ_1, λ_2 arbitrary,
- (iii) $C = -16$, $\lambda_1 = \lambda_2/16$.

These integrable cases correspond precisely to the stationary flows of the only three integrable cases of the fifth-order polynomial nonlinear evolution equations of scale weight 7 (respectively the Sawada–Kotega, the fifth-order Korteweg–de Vries and the Kaup–Kupershmidt equations) [8, 23].

In all above-mentioned cases system (2) is integrable at any value of μ . The function y , solution of system (2), satisfies the following fourth-order equation [10, 12, 23]:

$$y_{tttt} = (2C - 8)y_{tt}y - (4\lambda_1 + \lambda_2)y_{tt} + 2(C + 1)y_t^2 + \frac{20C}{3}y^3 + (4C\lambda_1 - 6\lambda_2)y^2 - 4\lambda_1\lambda_2y - 4H, \quad (3)$$

where H is the energy of the system. We note, that H is not an arbitrary parameter, but a function of initial data: y_0 , y_{0t} , y_{0tt} and y_{0ttt} . The form of this function depends on μ :

$$H = \frac{1}{2}(y_{0t}^2 + y_0^2) - \frac{C}{3}y_0^3 + \left(\frac{\lambda_1}{2} + y_0\right)(Cy_0^2 - \lambda_2y_0 - y_{0tt}) + \frac{(\lambda_2y_{0t} + 2Cy_0y_{0t} - y_{0ttt})^2 + \mu}{2(Cy_0^2 - \lambda_2y_0 - y_{0tt})}.$$

This formula is correct only if $x_0 = Cy_0^2 - \lambda_2y_0 - y_{0tt} \neq 0$. If $x_0 = 0$, what is possible only at $\mu = 0$, then we can not express x_{0t} through y_0 , y_{0t} , y_{0tt} and y_{0ttt} , so H is not a function of the

initial data. If $y_{0ttt} = 2Cy_0y_{0t} - \lambda_2y_{0t}$, then eq. (3) with an arbitrary H corresponds to system (2) with $\mu = 0$, in opposite case eq. (3) does not correspond to system (2).

To find a special solution of eq. (3) one can assume that y satisfies some more simple equation. For example, there exist solutions in terms of the Weierstrass elliptic functions, which satisfy the following equation:

$$y_t^2 = \mathcal{A}y^3 + \mathcal{B}y^2 + \mathcal{C}y + \mathcal{D}, \quad (4)$$

where \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} are some constants.

The following generalization of eq. (4):

$$y_t^2 = \tilde{\mathcal{A}}y^3 + \tilde{\mathcal{B}}y^{5/2} + \tilde{\mathcal{C}}y^2 + \tilde{\mathcal{D}}y^{3/2} + \tilde{\mathcal{E}}y + \tilde{\mathcal{G}} \quad (5)$$

gives new one-parameter solutions in two nonintegrable cases [12]: $C = -16/5$ and $C = -4/3$ (λ_1 is an arbitrary number, $\lambda_2 = 1$). It is easy to show [12] that if $\tilde{\mathcal{B}} \neq 0$ or $\tilde{\mathcal{D}} \neq 0$ then $\tilde{\mathcal{G}} = 0$, therefore, substitution $y = \varrho^2$ transforms eq. (4) into

$$\varrho_t^2 = \frac{1}{4}(\tilde{\mathcal{A}}\varrho^4 + \tilde{\mathcal{B}}\varrho^3 + \tilde{\mathcal{C}}\varrho^2 + \tilde{\mathcal{D}}\varrho + \tilde{\mathcal{E}}). \quad (6)$$

In [13] using the substitution $y \longrightarrow y - P_0$ a new parameter P_0 has been introduced and two-parameter solutions have been constructed for above-mentioned values of C and a few values of λ_1 ($\lambda_2 = 1$). Due to Painlevé analysis local three-parameter solutions as the converging Laurent series have been found for an arbitrary λ_1 , $\lambda_2 = 1$ and $\mu = 0$ [18]. In the present paper we seek both the elliptic and the Laurent-series solutions for arbitrary values of λ_1 , λ_2 and μ .

3 NEW SOLUTIONS

Let us assume that solutions of eq. (3) in the neighborhood of singularity point t_0 tend to infinity as $y = c_\beta(t - t_0)^\beta$, where β and c_β are some complex numbers. Of course, the real part of β has to be less than zero. From this assumption it follows [22] that $\beta = -2$. The Laurent series of solutions of eq. (6) begin with term proportional to $(t - t_0)^{-1}$, so we seek solutions of eq. (3) as square polynomial: $y = P_2\varrho^2 + P_1\varrho + P_0$, where P_2 , P_1 and P_0 are arbitrary numbers, ϱ is the general solution of eq. (6) with arbitrary coefficients $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$, $\tilde{\mathcal{C}}$, $\tilde{\mathcal{D}}$ and $\tilde{\mathcal{E}}$. Because of the function $\tilde{\varrho} = (\varrho - \frac{P_1}{2})/\sqrt{P_2}$ is a solution of eq. (6) as well, we can put $P_2 = 1$ and $P_1 = 0$ without loss the generality.

Substituting $y = \varrho^2 + P_0$ in eq. (3), we obtain

$$\begin{aligned} \varrho_{tttt}\varrho = & -4\varrho_{ttt}\varrho_t - 3\varrho_{tt}^2 + 2(C-4)\varrho_{tt}\varrho^3 + (2P_0(C-4) - 4\lambda_1 - \lambda_2)\varrho_{tt}\varrho + \\ & + 2(3C-2)\varrho_t^2\varrho^2 + (2CP_0 - 4\lambda_1 - 8P_0 - \lambda_2)\varrho_t^2 + \frac{10}{3}C\varrho^6 + \\ & + (2C\lambda_1 + 10CP_0 - 3\lambda_2)\varrho^4 + 2(2\lambda_1CP_0 + 5CP_0^2 - \lambda_1\lambda_2 - 3P_0\lambda_2)\varrho^2 + \\ & + \frac{10}{3}CP_0^3 + 2\lambda_1CP_0^2 - 3P_0^2\lambda_2 - 2\lambda_1\lambda_2P_0 - 2H. \end{aligned} \quad (7)$$

The function ϱ is a solution of eq. (6), hence, eq. (7) is equivalent to the following system:

$$\left\{ \begin{array}{l} (3\tilde{\mathcal{A}} + 4)(-3\tilde{\mathcal{A}} + 2C) = 0, \\ \tilde{\mathcal{B}}(-21\tilde{\mathcal{A}} + 9C - 16) = 0, \\ 96\tilde{\mathcal{A}}CP_0 - 240\tilde{\mathcal{A}}\tilde{\mathcal{C}} - 192\tilde{\mathcal{A}}\lambda_1 - 384\tilde{\mathcal{A}}P_0 - 48\tilde{\mathcal{A}}\lambda_2 - \\ - 105\tilde{\mathcal{B}}^2 + 128\tilde{\mathcal{C}}C - 192\tilde{\mathcal{C}} + 128C\lambda_1 + 640CP_0 - 192\lambda_2 = 0, \\ 40\tilde{\mathcal{B}}CP_0 - 90\tilde{\mathcal{A}}\tilde{\mathcal{D}} - 65\tilde{\mathcal{B}}\tilde{\mathcal{C}} - 80\tilde{\mathcal{B}}\lambda_1 - 160\tilde{\mathcal{B}}P_0 - 20\tilde{\mathcal{B}}\lambda_2 + 56C\tilde{\mathcal{D}} - 64\tilde{\mathcal{D}} = 0, \\ 16\tilde{\mathcal{C}}CP_0 - 36\tilde{\mathcal{A}}\tilde{\mathcal{E}} - 21\tilde{\mathcal{B}}\tilde{\mathcal{D}} - 8\tilde{\mathcal{C}}^2 - 32\tilde{\mathcal{C}}\lambda_1 - 64\tilde{\mathcal{C}}P_0 - 8\lambda_2\tilde{\mathcal{C}} + 24C\tilde{\mathcal{E}} + \\ + 64\lambda_1CP_0 + 160CP_0^2 - 16\tilde{\mathcal{E}} - 32\lambda_1\lambda_2 - 96P_0\lambda_2 = 0, \\ 10\tilde{\mathcal{B}}\tilde{\mathcal{E}} + (5\tilde{\mathcal{C}} + 8CP_0 - 16\lambda_1 - 32P_0 - 4\lambda_2)\tilde{\mathcal{D}} = 0, \\ 384H = -48\tilde{\mathcal{C}}\tilde{\mathcal{E}} + 96C\tilde{\mathcal{E}}P_0 + 384C\lambda_1P_0^2 + 640CP_0^3 - 9\tilde{\mathcal{D}}^2 - \\ - 192\tilde{\mathcal{E}}\lambda_1 - 384\tilde{\mathcal{E}}P_0 - 48\tilde{\mathcal{E}}\lambda_2 - 384\lambda_1\lambda_2P_0 - 576\lambda_2P_0^2. \end{array} \right. \quad (8)$$

System (8) has been solved by computer algebra software REDUCE [26].

If $\tilde{\mathcal{B}} \neq 0$, then from two first equations of system (8) we obtain:

$$C = -\frac{4}{3} \quad \text{and} \quad \tilde{\mathcal{A}} = -\frac{4}{3} \quad \text{or} \quad C = -\frac{16}{5} \quad \text{and} \quad \tilde{\mathcal{A}} = -\frac{32}{15}.$$

If $\tilde{\mathcal{B}} = 0$, then solutions with $\tilde{\mathcal{D}} \neq 0$ are also possible at $C = -16$ and $C = -1$, but only in integrable cases. The obtained solutions of eq. (3) depend on two parameters: energy H expressed through P_0 and parameter t_0 connected to homogeneity of time.

Six solutions of system (8) correspond to each value of P_0 . Two of them (with $\tilde{\mathcal{B}} = \tilde{\mathcal{D}} = 0$) generate solutions of eq. (4). Values of $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{D}}$, corresponding to other solutions, depend on λ_1 and λ_2 and are zero only at some relations between these parameters. We will consider only solutions with $\tilde{\mathcal{B}} \neq 0$ or $\tilde{\mathcal{D}} \neq 0$. They are presented in Appendix. These solutions can be separated on pairs in such a way that solutions in one pair differ only in signs of $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{D}}$. Basic properties of the obtained solution are considered in this section. In the next section we analyze in detail solutions of system (8) for some values of λ_1 and λ_2 .

If the right-hand side of eq. (6) is a polynomial with multiple roots, then ϱ and y can be expressed in terms of elementary functions. In opposite case y is an elliptic function [24, 25].

It is simplicity itself that $y(t) = \varrho^2(t) + P_0 = (-\varrho(t))^2 + P_0$, so, solutions of system (8) with opposite values of $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{D}}$ generate identical solutions of eq. (3). From eq. (6) we obtain a polynomial equation for $y(t)$:

$$(y_t^2 - \tilde{\mathcal{A}}(y - P_0)^3 - \tilde{\mathcal{C}}(y - P_0)^2 - \tilde{\mathcal{E}}(y - P_0))^2 = (y - P_0)^3(\tilde{\mathcal{B}}(y - P_0) + \tilde{\mathcal{D}})^2. \quad (9)$$

The function $\varrho(t)$ can be expressed through the Weierstrass elliptic function $\wp(t)$ [25, Ch. 5]:

$$\varrho(t - t_0) = \frac{a\wp(t - t_0) + b}{c\wp(t - t_0) + d}, \quad (ad - bc = 1),$$

where t_0 is an arbitrary parameter. Periods of $\wp(t)$ and the constants a, b, c and d are determined by eq. (6). The function

$$y(t - t_0) = \left(\frac{a\wp(t - t_0) + b}{c\wp(t - t_0) + d} \right)^2 + P_0 \quad (10)$$

is the fourth-order elliptic function. This function, as a solution of eq. (3), can have only the second-order poles, therefore, in the parallelogram of periods it has two poles with opposite residues. Solutions (10) differ from solutions of eq. (4), which are the second-order elliptic functions [25].

The function $x(t)$ satisfies the first equation of system (2) with

$$\begin{aligned} \mu = & \frac{8}{3}C^2P_0^5 + \left(2\lambda_1C^2 - \frac{14}{3}\lambda_2C\right)P_0^4 + \left(2\lambda_2^2 - \frac{10}{3}C\tilde{\mathcal{E}} - 4\lambda_1\lambda_2C\right)P_0^3 + \left(2\lambda_1\lambda_2^2 - \right. \\ & \left. - 2\lambda_1C\tilde{\mathcal{E}} - 4CH + 3\lambda_2\tilde{\mathcal{E}}\right)P_0^2 + \left(2\lambda_1\lambda_2\tilde{\mathcal{E}} + \tilde{\mathcal{E}}^2 + 4\lambda_2H\right)P_0 + 2\tilde{\mathcal{E}}H + \frac{1}{2}\lambda_1\tilde{\mathcal{E}}^2 + \frac{9}{128}\tilde{\mathcal{D}}^2\tilde{\mathcal{E}}. \end{aligned} \quad (11)$$

The trajectory of the motion can be derived from the second equation of system (2). Substituting y_{tt} , we obtain:

$$x^2 = \left(C - \frac{3}{2}\tilde{\mathcal{A}}\right)y^2 + (3\tilde{\mathcal{A}}P_0 - \tilde{\mathcal{C}} - 1)y - \frac{1}{4}(5\tilde{\mathcal{B}}y + 3\tilde{\mathcal{D}} - 5\tilde{\mathcal{B}}P_0)\sqrt{y - P_0} - \frac{1}{2}(\tilde{\mathcal{E}} + 3\tilde{\mathcal{A}}P_0^2 - 2\tilde{\mathcal{C}}P_0).$$

If $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{D}}$ take zero values we get simple algebraic trajectories. The full list of such trajectories is presented in [11]. The parameter P_0 is absent in these trajectory equations.

One value of the energy H can correspond to no more than three values of P_0 and, hence, no more than six different one-parameter solutions. Solutions (10) differ from solutions of eq. (4), which are the second-order elliptic functions [25].

4 A PARTICULAR CASE

4.1 The form of solutions

At $C = -16/5$, $\lambda_1 = 1/9$ and $\lambda_2 = 1$ one-parameter solutions ($P_0 = 0$) have been considered in detail in our previous papers [12, 18]. For these values of parameters solutions of system (8) are:

$$\begin{aligned} 1. \quad & \tilde{\mathcal{A}} = -\frac{32}{15}, \quad \tilde{\mathcal{B}} = 0, & \tilde{\mathcal{C}} = -\frac{32}{5}P_0 - 1, \\ & \tilde{\mathcal{D}} = 0, \quad \tilde{\mathcal{E}} = -\frac{32}{5}P_0^2 - 2P_0, & H = \frac{16}{15}P_0^3 + \frac{1}{2}P_0^2 \\ 2. \quad & \tilde{\mathcal{A}} = -\frac{4}{3}, \quad \tilde{\mathcal{B}} = 0, & \tilde{\mathcal{C}} = -4P_0 - \frac{17}{33}, \\ & \tilde{\mathcal{D}} = 0, \quad \tilde{\mathcal{E}} = -4P_0^2 - \frac{34}{33}P_0 + \frac{20}{3267}, & H = -\frac{2}{15}P_0^3 - \frac{17}{330}P_0^2 + \frac{2}{3267}P_0 - \frac{230}{323433}, \\ 3-4. \quad & \tilde{\mathcal{A}} = -\frac{32}{15}, & \tilde{\mathcal{B}} = \pm \frac{8i\sqrt{15}}{45}, & \tilde{\mathcal{C}} = -\frac{32}{5}P_0 - \frac{4}{9}, \\ & \tilde{\mathcal{D}} = \pm \frac{4i\sqrt{15}}{9}P_0, & \tilde{\mathcal{E}} = -\frac{32}{5}P_0^2 - \frac{8}{9}P_0, & H = \frac{16}{15}P_0^3 - \frac{7}{72}P_0^2, \end{aligned}$$

$$\begin{aligned}
5-6. \quad \tilde{\mathcal{A}} &= -\frac{32}{15}, & \tilde{\mathcal{D}} &= \pm \frac{\sqrt{65}\sqrt{561}}{11329956}(26928P_0 + 8125), \\
\tilde{\mathcal{B}} &= \pm \frac{8}{8415}\sqrt{65}\sqrt{561}, & \tilde{\mathcal{E}} &= -\frac{32}{5}P_0^2 - \frac{3496}{1683}P_0 - \frac{333125}{7553304}, \\
\tilde{\mathcal{C}} &= -\frac{32}{5}P_0 - \frac{1748}{1683}, & H &= \frac{16}{15}P_0^3 + \frac{7291}{13464}P_0^2 + \frac{6426875}{181279296}P_0 + \frac{17551324375}{9762977765376}.
\end{aligned}$$

If the right-hand side of eq. (6) is a polynomial with multiple roots, then the function y can be expressed in terms of elementary functions. For example, at $P_0 = 0$ substitution of solutions 3-4 into eq. (5) gives

$$y = -\frac{5}{3\left(1 - 3\sin\left(\frac{t-t_0}{3}\right)\right)^2}, \quad (12)$$

where t_0 is an arbitrary constant.

From (11) we obtain the following values of μ :

1. $\mu = 0,$
2. $\mu = \frac{160}{1089}P_0^3 + \frac{680}{11979}P_0^2 - \frac{800}{1185921}P_0 - \frac{7000}{1056655611},$
- 3-4. $\mu = \frac{4}{3}P_0^4 + \frac{5}{54}P_0^3 + \frac{50}{729}P_0^2,$
- 5-6. $\mu = -\frac{52}{561}P_0^4 - \frac{81640}{944163}P_0^3 - \frac{4458460825}{152546527584}P_0^2 - \frac{539878421875}{128367902961936}P_0 - \frac{728473377734375}{6703885364284145664}.$

4.2 Motion trajectories

Let us consider the equations of the motion trajectories at $C = -16/5$ and $\lambda = 1/9$. In the case of the solutions with $\tilde{\mathcal{B}} = \tilde{\mathcal{D}} = 0$ the trajectory equation can be reduced either to $x^2 = 0$ (solution 1), or to

$$x^2 + \frac{6}{5}\left(y + \frac{20}{99}\right)^2 = \frac{50}{1089}. \quad (13)$$

In the last case (solution 2) the motion trajectory is an ellipse. Note, however, that the real motion does not necessarily affect the whole ellipse: it depends on two arbitrary parameters. The energy H can be considered as one of them.

In the case of solutions 3-4 the trajectory equation is the following:

$$\left(x^2 + \frac{5}{9}y\right)^2 + \frac{5}{27}(y - P_0)(2y + P_0)^2 = 0. \quad (14)$$

If $P_0 = 0$ (see (12)), the equation for one of the trajectory branches entirely coincides with the equation obtained in [12]. The condition $y < 0$ is always required for the existence of real motion along these trajectories. Formula (9) describes precisely such a solution. For solutions 5-6 the trajectory equation has the same form as for solutions 3-4.

5 THREE-PARAMETER SOLUTIONS

The Ablowitz–Ramani–Segur algorithm of the Painlevé test [20] is very useful for obtaining the solutions as formal Laurent series. Let the behavior of a solution in the neighborhood of the

singularity point t_0 be algebraic, i.e., x and y tend to infinity as some powers: $x = a_\alpha(t - t_0)^\alpha$ and $y = b_\beta(t - t_0)^\beta$, where α , β , a_α and b_β are some constants. If α and β are negative integer numbers, then substituting the Laurent series expansions one can transform nonlinear differential equations into a system of linear algebraic equations on coefficients of Laurent series. If a single-valued solution depends on more than one arbitrary parameters then some coefficients of its Laurent series have to be arbitrary and the corresponding systems have to have zero determinants. The numbers of such systems (named *resonances* or *Kovalevskaya exponents*) can be determined due to the Painlevé test.

Two possible dominant behaviors and resonance structures of solutions of the generalized Hénon–Heiles system [22, 27] and eq. (3) are presented in the Table.

<i>Case 1</i>	<i>Case 2: $\beta < \Re(\alpha)$</i>
$\alpha = -2,$ $\beta = -2,$ $a_\alpha = \pm 3\sqrt{2+C},$ $b_\beta = -3,$ $r = -1, 6, \frac{5}{2} \pm \frac{\sqrt{1-24(1+C)}}{2}$ $r_4 = -1, 10, \frac{5}{2} \pm \frac{\sqrt{1-24(1+C)}}{2}$	$\alpha = \frac{1 \pm \sqrt{1-48/C}}{2},$ $\beta = -2,$ $a_\alpha = c_1$ (arbitrary), $b_\beta = \frac{6}{C},$ $r = -1, 0, 6, \mp \sqrt{1 - \frac{48}{C}}$ $r_4 = -1, 5, 5 - \sqrt{1 - \frac{48}{C}}, 5 + \sqrt{1 - \frac{48}{C}}$

The values of r denote resonances: $r = -1$ corresponds to arbitrary parameter t_0 ; $r = 0$ (in the *Case 2*) corresponds to arbitrary parameter c_1 . Other values of r determine powers of t , to be exact, $t^{\alpha+r}$ for x and $t^{\beta+r}$ for y , at which new arbitrary parameters can appear as solutions of the linear systems with zero determinant. Note, that the dominant behaviour and the resonance structure depend only on C .

It is necessary for the integrability of system (2) that all values of r be integer and that all systems with zero determinants have solutions for any values of the free parameters entering these systems. This is possible only in the integrable cases (i)–(iii).

For the search for special solutions, it is interesting to consider such values of C , for which r are integer numbers either only in *Case 1* or only in *Case 2*. If there exist a negative integer resonance, different from $r = -1$, then such Laurent series expansion corresponds rather to special than general solution [22]. We demand that all values of r , but one, are nonnegative integer numbers and all these values are different. From these conditions we obtain the following values of C : $C = -1$ and $C = -4/3$ (*Case 1*), or $C = -16/5$, $C = -6$ and $C = -16$ (*Case 2*, $\alpha = \frac{1-\sqrt{1-48/C}}{2}$), and also $C = -2$, in which these two *Cases* coincide. It is remarkable that only for these values of C there exist solutions of system (8) with $\tilde{\mathcal{B}} \neq 0$ or $\tilde{\mathcal{D}} \neq 0$.

Let us consider the possibility of existence of the single-valued three-parameter solutions in all these cases. To obtain the result for an arbitrary value of μ , we consider eq. (3) with an

arbitrary H . Note, that the values of resonances obtained from eq. (3) (in the Table they are signified as r_4) are different from r , but we obtain the same result: condition that all values of r_4 , but $r_4 = -1$, are nonnegative integer numbers gives the same values of C .

At $C = -2$ we have a contradiction: $r_4 = 0$, but b_{-2} is not arbitrary parameter: $b_{-2} = -3$. This is the consequence of the fact that, contrary to our assumption, the behaviour of the general solution in the neighborhood of a singular point is not algebraic, because its dominant term includes logarithm [22]. At $C = -6$ and any value of other parameters the exact four-parameter solutions are known. In cases $C = -1$ and $C = -16$ the substitution of an unknown function as the Laurent series leads to the conditions $\lambda_1 = \lambda_2$ or $\lambda_1 = \lambda_2/16$ accordingly. Hence, in nonintegrable cases three-parameter local solutions have to include logarithmic terms. Single-valued three-parameter solutions can exist only in two above-mentioned nonintegrable cases: $C = -16/5$ and $C = -4/3$.

Using the method of construction of the Laurent series solutions for nonlinear differential equations describing in [18], we obtain single-valued local solutions of eq. (3) both at $C = -16/5$ and at $C = -4/3$. Values of other parameters are arbitrary.

At $C = -4/3$ these solutions are:

$$y = -3\frac{1}{t^2} + b_{-1}\frac{1}{t} + \frac{29}{24}b_{-1}^2 + \frac{1}{2}\lambda_1 - \frac{3}{4}\lambda_2 + \left(\frac{17}{6}b_{-1}^2 + \frac{5}{3}\lambda_1 - \frac{5}{4}\lambda_2\right)b_{-1}t + b_2t^2 - \left(\frac{55}{12}\lambda_1b_{-1}^2 + \frac{131}{90}\lambda_1^2 + \frac{33}{40}\lambda_2^2 + \frac{9359}{2592}b_{-1}^4 + b_2 - \frac{55}{16}\lambda_2b_{-1}^2 - \frac{131}{60}\lambda_1\lambda_2\right)b_{-1}^2t^3 + \dots \quad (15)$$

There exist four possible values of the parameter b_{-1} :

$$b_{-1} = \pm \sqrt{\frac{105\lambda_2 - 140\lambda_1 + \sqrt{7(1216\lambda_1^2 - 1824\lambda_1\lambda_2 + 783\lambda_2^2)}}{385}}$$

or

$$b_{-1} = \pm \sqrt{\frac{105\lambda_2 - 140\lambda_1 - \sqrt{7(1216\lambda_1^2 - 1824\lambda_1\lambda_2 + 783\lambda_2^2)}}{385}}.$$

The parameters b_2 and b_8 , coefficients at t^2 and t^8 correspondingly, are arbitrary. The energy H enters in coefficients beginning from b_4 .

At $C = -16/5$ we obtain the following solutions:

$$y = -\frac{15}{8t^{-2}} + \tilde{b}_{-1} - \frac{5}{32}\lambda_2 + \frac{62}{45}\tilde{b}_{-1}^2 + \left(\frac{5}{12}\lambda_1 + \frac{632}{225}\tilde{b}_{-1}^2 - \frac{25}{192}\lambda_2\right)\tilde{b}_{-1}t + \left(\frac{29}{15}\lambda_1\tilde{b}_{-1}^2 - \frac{1}{128}\lambda_2^2 - \frac{29}{48}\lambda_2\tilde{b}_{-1}^2 + \frac{102272}{10125}\tilde{b}_{-1}^4\right)t^2 + \tilde{b}_3t^3 + \dots, \quad (16)$$

with

$$\tilde{b}_{-1} = \frac{\pm 3}{41888} \sqrt{6872250\lambda_2 - 21991200\lambda_1 + 52360\sqrt{71680\lambda_1^2 - 44800\lambda_1\lambda_2 + 13545\lambda_2^2}}$$

or

$$\tilde{b}_{-1} = \frac{\pm 3}{41888} \sqrt{6872250\lambda_2 - 21991200\lambda_1 - 52360\sqrt{71680\lambda_1^2 - 44800\lambda_1\lambda_2 + 13545\lambda_2^2}}.$$

The coefficients \tilde{b}_3 and \tilde{b}_8 are arbitrary parameters. Beginning from \tilde{b}_4 some coefficients include the energy H . So, the obtained local solutions depend on four independent parameters: t_0 , H and two coefficients (b_2 and b_8 or \tilde{b}_3 and \tilde{b}_8).

We have found local single-valued solutions. Of course, existence of local single-valued solutions is necessary, but not sufficient condition to exist global ones, because solutions, which are single-valued in the neighborhood of one singularity point, can be multivalued in the neighborhood of another singularity point. So, we can only assume that global three-parameter solutions are single-valued. If we assume this and moreover that these solutions are elliptic functions (or some degenerations of them), then we can seek them as solutions of some polynomial first order equations. There are a few methods to construct such solutions [8, 10, 28, 29]. Using these methods one represents a solution of a nonlinear ordinary differential equation (ODE) as the finite Taylor or Laurent series of elliptic functions or degenerate elliptic functions, for example, $\tanh(t)$. Similar method is applied in this paper to find two-parameter solutions. These methods use results of the Painlevé test, but don't use the obtained Laurent-series solutions. In 2003 R. Conte and M. Musette [30] have proposed the method, which uses such solutions.

The classical theorem, which was established by Briot and Bouquet [31], proves that if the general solution of a polynomial autonomous first order ODE is single-valued, then this solution is either an elliptic function, or a rational function of $e^{\gamma x}$, γ being some constant, or a rational function of x . Note that the third case is a degeneracy of the second one, which in its turn is a degeneracy of the first one. It has been proved by Painlevé [19] that the necessary form of the polynomial autonomous first order ODE with the single-valued general solution is

$$\sum_{k=0}^m \sum_{j=0}^{2m-2k} h_{jk} y^j y_t^k = 0, \quad h_{0m} = 1, \quad (17)$$

in which m is a positive integer number and h_{jk} are constants.

Rather than to substitute eq. (17) in some nonintegrable system, one can substitute the Laurent series of unknown special solutions, for example, (15) or (16) in eq. (17) and obtain a system, which is linear in h_{jk} and nonlinear in the parameters, including in the Laurent coefficients [30]. There are a few computer algebra algorithms which allow to obtain this system from the given Laurent series. Moreover it is possible to exclude all h_{jk} from this system and obtain a nonlinear system in parameters of nonintegrable system and free parameters from the Laurent series. The main preference of this method is that the number of unknowns in the resulting nonlinear algebraic system does not depend on number of coefficients of the first order equation. For example, eq. (17) with $m = 8$ includes 60 unknowns h_{jk} , and it is not possible use the traditional way to find similar solutions. Using this method we always obtain nonlinear system in 5 variables: λ_1 , λ_2 , H and two arbitrary coefficients of the Laurent-series solutions. We hope that this method allows us to find three-parameter global solutions.

6 Conclusions

Two nonintegrable cases ($C = -16/5$ or $C = -4/3$, λ_1 , λ_2 and μ are arbitrary) of the generalized Hénon–Heiles system with the nonpolynomial term have been considered. To avoid problems with the nonpolynomial term we have transformed system into the fourth-order equation. Two-parameter elliptic solutions for this equation have been found in both above-mentioned

cases. Two different solutions correspond to each pair of parameter values. The Painlevé test does not show any obstacle to the existence of three-parameter single-valued solutions, so, the probability to find exact, for example elliptic, three-parameter solutions, that generalize the obtained solutions, is high.

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APPENDIX

In two nonintegrable cases ($C = -16/5$ and $C = -4/3$) for arbitrary λ_1 and λ_2 we obtain that six solutions of system (8) correspond to each value of P_0 . Two of them (with $\tilde{\mathcal{B}} = \tilde{\mathcal{D}} = 0$) generate solutions of eq. (4). Other solutions of system (8) can be separated on pairs such as each pair of solutions corresponds to one two-parameter function $y = \varrho^2 + P_0$, where ϱ satisfies eq. (6) with the following values of coefficients:

$$C = -\frac{16}{5},$$

$$\tilde{\mathcal{A}} = -\frac{32}{15},$$

$$\tilde{\mathcal{B}} = -\frac{\sqrt{1122}(1120\lambda_1 + 41888P_0 + 65S_q + 6195\lambda_2)\sqrt{F_1(\lambda_1, \lambda_2, P_0)}}{29373960(3600\lambda_1^2 - 1120\lambda_1P_0 - 2425\lambda_1\lambda_2 - 20944P_0^2 - 6195\lambda_2P_0 + 225\lambda_2^2)},$$

$$\tilde{\mathcal{C}} = -\frac{240}{187}\lambda_1 - \frac{32}{5}P_0 + \frac{4}{1309}S_q - \frac{112}{187}\lambda_2,$$

$$\tilde{\mathcal{D}} = \frac{\sqrt{1122}}{5874792}\sqrt{F_1(\lambda_1, \lambda_2, P_0)},$$

$$\begin{aligned} \tilde{\mathcal{E}} = & \frac{88320}{244783}\lambda_1^2 - \frac{480}{187}\lambda_1P_0 + \frac{885}{244783}\lambda_1S_q - \frac{153375}{244783}\lambda_1\lambda_2 - \\ & - \frac{32}{5}P_0^2 + \frac{8}{1309}P_0S_q - \frac{224}{187}\lambda_2P_0 - \frac{685}{3916528}\lambda_2S_q + \frac{168855}{3916528}\lambda_2^2, \end{aligned}$$

$$\begin{aligned} H = & -\frac{11516270}{45774421}\lambda_1^3 + \frac{8740}{34969}\lambda_1^2P_0 - \frac{3296515}{2563367576}\lambda_1^2S_q + \frac{50336425}{183097684}\lambda_1^2\lambda_2 + \\ & + \frac{258}{187}\lambda_1P_0^2 - \frac{8209}{1958264}\lambda_1P_0S_q + \frac{76915}{279752}\lambda_1\lambda_2P_0 + \frac{12202395}{82027762432}\lambda_1\lambda_2S_q - \\ & - \frac{131879855}{11718251776}\lambda_1\lambda_2^2 - \frac{43}{13090}P_0^2S_q + \frac{103}{1496}\lambda_2P_0^2 + \frac{8881}{31332224}\lambda_2P_0S_q - \\ & - \frac{71205}{4476032}\lambda_2^2P_0 - \frac{12990165}{1312444198912}\lambda_2^2S_q - \frac{168661575}{187492028416}\lambda_2^3 + \frac{16}{15}P_0^3, \end{aligned}$$

$$C = -\frac{4}{3}, \quad \tilde{\mathcal{A}} = -\frac{4}{3},$$

$$\tilde{\mathcal{B}} = \frac{\sqrt{330}(952\lambda_1 - 616P_0 + 13R_q - 945\lambda_2)\sqrt{F_2(\lambda_1, \lambda_2, P_0)}}{38115(432\lambda_1^2 + 952\lambda_1P_0 - 291\lambda_1\lambda_2 - 308P_0^2 - 945P_0\lambda_2 + 27\lambda_2^2)},$$

$$\tilde{\mathcal{C}} = -\frac{4}{33}\lambda_1 - 4P_0 - \frac{1}{66}R_q - \frac{31}{22}\lambda_2,$$

$$\tilde{\mathcal{D}} = \frac{\sqrt{330}}{7623}\sqrt{F_2(\lambda_1, \lambda_2, P_0)},$$

$$\begin{aligned} \tilde{\mathcal{E}} = & \frac{3394}{363}\lambda_1^2 + \frac{54}{11}\lambda_1P_0 - \frac{1123}{10164}\lambda_1R_q - \frac{5897}{484}\lambda_1\lambda_2 - \\ & - \frac{17}{3}P_0^2 - \frac{31}{308}P_0R_q - \frac{349}{44}\lambda_2P_0 + \frac{1223}{27104}\lambda_2R_q + \frac{13005}{3872}\lambda_2^2, \end{aligned}$$

$$\begin{aligned} H = & -\frac{552922}{83853}\lambda_1^3 - \frac{29801}{2541}\lambda_1^2P_0 + \frac{173605}{2347884}\lambda_1^2R_q + \frac{778033}{74536}\lambda_1^2\lambda_2 - \frac{185}{66}\lambda_1P_0^2 + \\ & + \frac{3001}{20328}\lambda_1P_0R_q + \frac{104959}{6776}\lambda_1\lambda_2P_0 - \frac{695609}{12522048}\lambda_1\lambda_2R_q - \frac{2990049}{596288}\lambda_1\lambda_2^2 + \frac{89}{1232}P_0^2R_q + \\ & + \frac{5}{2}P_0^3 + \frac{865}{176}\lambda_2P_0^2 - \frac{3065}{54208}\lambda_2P_0R_q - \frac{225909}{54208}\lambda_2^2P_0 + \frac{2733}{260876}\lambda_2^2R_q + \frac{57699}{74536}\lambda_2^3, \end{aligned}$$

where

$$\begin{aligned} F_1(\lambda_1, \lambda_2, P_0) \equiv & 39474176000\lambda_1^3 + 122782105600\lambda_1^2P_0 - 104358400\lambda_1^2S_q - \\ & - 17822336000\lambda_1^2\lambda_2 + 210552545280\lambda_1P_0^2 - 680261120\lambda_1P_0S_q - 10941145600\lambda_1\lambda_2P_0 - \\ & - 41066800\lambda_1\lambda_2S_q + 8305290000\lambda_1\lambda_2^2 - 501315584P_0^2S_q - 65797670400\lambda_2P_0^2 + \\ & + 55920480P_0S_q + 1611640800\lambda_2^2P_0 + 2884725\lambda_2^2S_q - 468507375\lambda_2^3, \end{aligned}$$

$$S_q \equiv \pm\sqrt{35(2048\lambda_1^2 - 1280\lambda_1\lambda_2 + 387\lambda_2^2)},$$

$$\begin{aligned} F_2(\lambda_1, \lambda_2, P_0) \equiv & 2099776\lambda_1^3 - 497728\lambda_1^2P_0 - 20008\lambda_1^2R_q - 4911144\lambda_1^2\lambda_2 + 948640\lambda_1P_0^2 + \\ & + 19096\lambda_1P_0R_q + 1458072\lambda_1\lambda_2P_0 + 37173\lambda_1\lambda_2R_q + 3943233\lambda_1\lambda_2^2 + 6776P_0^2R_q - \\ & - 711480\lambda_2P_0^2 - 9240\lambda_2P_0R_q - 615384\lambda_2^2P_0 - 13581\lambda_2^2R_q - 1006425\lambda_2^3, \end{aligned}$$

$$R_q \equiv \pm\sqrt{7(1216\lambda_1^2 - 1824\lambda_1\lambda_2 + 783\lambda_2^2)}.$$

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